Lecture 21

Measure and Integration

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$$N = \{ x \in X \mid |f_n(x)| > g(x) \}$$
$$\cup \{ x \in X \mid f_n(x) \not\to f(x) \}$$

<u>On N^c</u> $\left. \begin{array}{l} |f_n(x)| < g(x) \\ f_n(x) \longrightarrow f(x) \end{array} \right\} \forall \; x \in N^c$

and $\mu(N) = 0$

<u>Consider</u> $\chi_{N^c} f_n , \; n \geq 1$

$$\left. \begin{array}{l} |\chi_{N^c} f_n| < g \\ \chi_{N^c} f_n \longrightarrow \chi_{N^c} f \end{array} \right|$$

$\chi_{N^c} f \in L_1$ and

$$\lim_{n\to\infty} \int \chi_{N^c} f_n \, d\mu \longrightarrow \int \chi_{N^c} f \, d\mu =$$

Note $\mu(N) = 0$

$$\Rightarrow \int_N f \, d\mu = 0$$

$$\Rightarrow \int |f| \, d\mu < +\infty \Rightarrow f \in L_1$$

$$\int f_n \, d\mu \longrightarrow \int f \, d\mu .$$

$$|f_n - f| \le 2g$$

$$|f_n - f| \longrightarrow 0$$

$$\underset{\Longrightarrow}{\text{DCTm}} \quad \int |f_n - f| \, d\mu \longrightarrow 0$$

We show ~~that~~

$$\sum_{n=1}^{\infty} f_n(x) \text{ is absolutely cgt.}$$

Define $$g_n(x) = \sum_{k=1}^{n} |f_k(x)|$$

N.B. $$g_n \uparrow \sum_{k=1}^{\infty} |f_k(x)| := g(x)$$

M.C Thm $\Rightarrow$ $$\int g_n \, d\mu \longrightarrow \int g \, d\mu$$

$\Rightarrow$ $$\int g \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int |f_k| \, d\mu$$

$$= \sum_{k=1}^{\infty} \int |f_k| \, d\mu < +\infty$$

Thus $g \in L_1$

$$\Longrightarrow 0 \leq g(x) < +\infty \text{ a.e.}$$

Hence $\sum_{k=1}^{\infty} |f_k(x)| < +\infty$ a.e. (x)

$$\Longrightarrow f(x) := \sum_{k=1}^{\infty} f_k(x) < +\infty \text{ a.e. } (x)$$

Note $f(x) = \lim_{n \to \infty} \underbrace{\sum_{k=1}^{n} f_k(x)}_{= \varphi_n}$

$$|\varphi_n| = \left| \sum_{1}^{n} f_k(x) \right| \leq \sum_{1}^{n} |f_k| = g_n \leq g$$

$$\left\{\begin{array}{l} |\phi_n| \le g \\ \phi_n \longrightarrow f \text{ a.e.} \end{array}\right.$$

$$\overset{\text{DCTh}}{\Longrightarrow} \quad \int \phi_n \, d\mu \longrightarrow \int f \, d\mu$$

$$\begin{array}{c} \| \\ \displaystyle\sum_{k=1}^{n} \int f_k \, d\mu \end{array} \longrightarrow \int f \, d\mu$$

$$\int f \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu$$

$\blacksquare$

$$|f_n(x)| \leq \underline{M} \text{ for a.e } x.$$

$$g(x) = M \quad \forall x \in X.$$

N.G $\quad \int g \, d\mu = \int M \, d\mu = M\mu(X) < +\infty$

$$\Rightarrow g \in L_1$$

$$f_n(x) \longrightarrow f(x) \text{ a.e}$$

<u>DCThm</u> $\Rightarrow \quad \int f \, d\mu = \underline{\lim_{n\to\infty} \int f_n \, d\mu}$

<u>To show</u> $\forall\, g \in L_1(X, \mathcal{S}, \nu)$

$$\left.\begin{aligned} \int g\, d\nu &= \int f g\, d\mu, \\ \text{where} \quad \nu(E) &= \int_E f\, d\mu \end{aligned}\right] \quad (*)$$

$\boxed{\text{Step 1}}$ $\boxed{g = \chi_E, \ E \in \mathcal{S}}$

Then $\displaystyle \int g\, d\nu = \nu(E) = \int \chi_E f\, d\mu$

$\displaystyle = \int g f\, d\mu$

$\boxed{\text{Step 2}}$ $\displaystyle g = \sum_{i=1}^{n} a_i \chi_{E_i}, \ E_i \in \mathcal{S}$

$$\int g\,d\nu = \int \left(\sum_{i=1}^{n} a_i \chi_{E_i} \right) d\nu$$

$$= \sum_{i=1}^{n} a_i\, \nu(E_i)$$

$$= \sum_{i=1}^{n} a_i \left(\int \chi_{E_i} f\, d\mu \right)$$

$$= \int \underbrace{\sum_{i=1}^{n} (a_i \chi_{E_i})} f\, d\mu$$

$$= \int g f\, d\mu$$

(*) holds for non-negative simple fns g

Step 3

Let $g : X \longrightarrow [0, +\infty]$ be measurable.

$\Longrightarrow \exists \{s_n\}_{n \geq 1}$ of non-negative simple measurable fns s.t.

$$s_n \uparrow g.$$

By the Monotone Convergence Thm

$$\int g \, d\nu = \lim_{n \to \infty} \int s_n \, d\nu$$

Step 2

$$= \lim_{n \to \infty} \int s_n f \, d\mu$$

$$\overset{\text{MC Thm}}{=} \int g f \, d\mu$$

$\Longrightarrow$ (*) holds for non-negative mbl fns.

$\boxed{\text{Step 3}}$ Let $g \in L_1(X, \mathcal{S}, \nu)$

$g = g^+ - g^-$, $g^+ \in L_1(\nu)$, $g^- \in L_1(\nu)$

(Step 2) $\cancel{\int g^+ d\mu = \int g^+}$

$\int g^+ d\nu = \int g^+ f \, d\mu < +\infty$

and

$\int g^- d\nu = \int g^- f \, d\mu < +\infty$

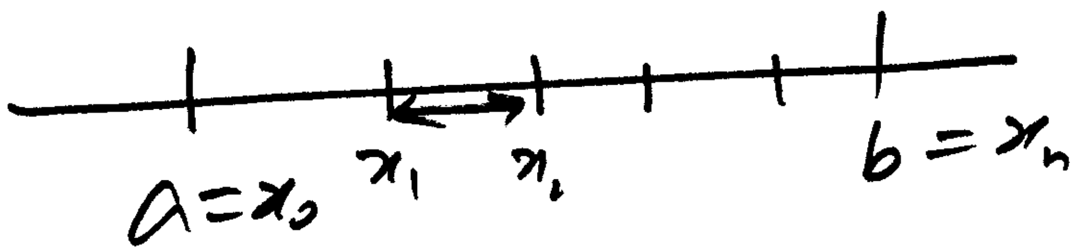
$\Rightarrow \int g f \, d\mu = \int g^+ f \, d\mu - \int g^- f \, d\mu$

$\Rightarrow (gf) \in L_1$, $= \int g \, d\nu$

Pf: $f \in \mathcal{R}[a,b]$ ||

$\Longrightarrow \exists \{P_n\}_{n\geq 1}$ of refinement partitions, $\|P_n\| \to 0$ as $n \to \infty$, with $\lim\limits_{n\to\infty} U(P_n,f) = \int_a^b f(x)\,dx = \lim\limits_{n\to\infty} L(P_n,f)$

$$P_n = \{a = x_0 < x_1 < \cdots < x_n = b\}$$



$$M_k = \max \text{ in } (x_{k-1}, x_k] \not{F} f(x)$$

$$m_k = \min_{(x_{n-1}, x_n]} \{ f(x) \}$$

$$\phi_k \cancel{(P_n, f)} = \sum M_k \chi_{(x_{n-1}, x_n]}$$

$$\psi_n \cancel{(P_n, f)} = \sum m_k \chi_{(x_{k-1}, x_k]}$$

$$U(P_n, f) = \int_a^b \phi_k(x)\, dx$$

$$L(P_n, f) = \int_a^b \psi_n(x)\, dx$$

Note $\quad \phi_k, \psi_k$ are measurable m

function

$$\phi_k(x) \geq f(x) \geq \psi_k(x)$$

and $\quad U(P_k, f) \geq \int_a^b f(x)\,dx \geq L(P_k, f)$

$$U(P_k, f) = \sum M_k (x_n - x_{n-1})$$
$$= \sum M_k \lambda(x_{n-1}, x_k]$$
$$= \int \phi_k \, d\lambda$$
$$L(P_k, f) = \int \psi_k \, d\lambda.$$

Consider $\phi_k - \psi_k \geq 0 \ \forall k$

and

$$\int (\phi_k - \psi_k)\, d\lambda \longrightarrow 0$$

$$\Rightarrow \quad \lim \phi_k(x) = \lim \psi_k(x) \quad a.e.$$

$$\left[\int \liminf (\phi_k - \psi_k)\, d\lambda \leq \liminf \underbrace{\int (\phi_k - \psi_k)}_{=0} \right.$$

$$\lim \phi_k(x) = f(x) = \lim \psi_k(x) \quad a.e.\ x$$

$\Rightarrow$ f is mble